A Brumer-Stark Conjecture for non-abelian Galois extensions

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Abstract. The Brumer-Stark conjecture deals with abelian extensions of number fields and predicts that a group ring element, called the Brumer-Stickelberger element constructed from special values of L-functions associated to the extension, annihilates the ideal class group of the extension under consideration. Moreover it specifies that the generators obtained have special properties. The aim of this article is to propose a generalization of this conjecture to non-abelian Galois extensions that is, in spirit, very similar to the original conjecture.

1. Introduction

The Brumer-Stark conjecture was first stated by Tate [13] and applies to abelian extensions of number fields. It combines a conjecture of Brumer that a certain group-ring element with integer coefficients constructed from the special values of L-functions associated the extension, the Brumer-Stickelberger element, annihilates the class group of the extension, with ideas coming from conjectures of Stark that predict special properties for a generator of the principal ideals obtained. A very nice reference for the Brumer-Stark conjecture, and Stark conjectures in general, is the book of Tate [14]. The aim of this article is to generalize the Brumer-Stark conjecture to Galois non-abelian extensions.

The plan of this paper is the following. In the second section, we state the Brumer-Stark conjecture, some of its properties and say a few words about its current status. To avoid confusion in the set of the paper, we will call this conjecture the abelian Brumer-Stark conjecture and will call the conjecture that we propose the Galois Brumer-Stark conjecture. The third section is devoted to the generalization of the Brumer-Stickelberger element to the Galois case. There, we rely on an earlier work of Hayes [9] that constructs this generalization and studies its properties. We show that it also satisfies additional properties very similar to the abelian case and, in particular, that it is rational. We are not able however to prove a suitable denominator for the Brumer-Stickelberger element, but we make a conjecture, called the Integrality Conjecture, of its value and makes this conjecture part of our generalization of the abelian Brumer-Stark conjecture. The next section introduces the notion of strong central extensions. This notion plays a fundamental part in our generalization. The Galois Brumer-Stark conjecture is stated in Section 5 and we study its properties in Section 6 with in view the generalization of the properties of the abelian Brumer-Stark conjecture. The last section is devoted to the study of the conjecture in the special case where the Galois group of the extension contains an abelian normal subgroup of prime index. In this setting, we prove that the abelian Brumer-Stark conjecture implies the Galois Brumer-Stark conjecture.

Note that generalizations to the non-abelian case of the Brumer-Stark conjecture, and other Stark conjectures in general, are also proposed in [2] and [10]. However, the direction of the generalizations, the points of view and the methods used are quite different from the ones we use here.

Convention. We denote the action of elements of Galois groups on elements, ideals, etc., using the exponent notation with the convention that they act on the left, that is $\alpha^{\sigma\gamma} = \sigma(\gamma(\alpha))$.

2. The abelian Brumer-Stark conjecture

In this section, we state the abelian Brumer-Stark conjecture and review some of its properties. Let K/k be an abelian extension of number fields. Denote by G its Galois group. Fix S a finite set of places of k containing the infinite places of k and the finite places of k that ramify in K/k. We assume that the cardinality of S is at least two. To a character χ of G is associated the S-truncated Hecke L-function of χ defined for $\mathrm{Re}(s) > 1$ by

$$L_{K/k,S}(s,\chi) := \prod_{\mathfrak{p} \notin S} (1 - \chi(\sigma_{\mathfrak{p}})\mathcal{N}(\mathfrak{p})^{-s})^{-1}$$

where $\mathfrak{p}$ runs through the prime ideals of k not in S, $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism of $\mathfrak{p}$ in G, and $\mathcal{N}(\mathfrak{p})$ is the absolute norm of the ideal $\mathfrak{p}$. These functions admit meromorphic continuation to $\mathbb{C}$, and in fact analytic if the character χ is non-trivial. A main object of the abelian Brumer-Stark conjecture is the Brumer-Stickelberger element which is constructed from the values at $s = 0$ of Hecke L-functions. It is a relative analogue of the Stickelberger element of cyclotomic fields and is defined by the formula

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0,\chi)\, e_{\bar{\chi}} \in \mathbb{C}[G]$$

where $\hat{G}$ denotes the group of characters of G and, for $\chi \in \hat{G}$, e_χ is the associated idempotent. Another characterization of this element is to say that it is the only element in $\mathbb{C}[G]$ such that

$$\chi(\theta_{K/k,S}) = L_{K/k,S}(0,\bar{\chi}) \tag{2.1}$$

for all character $\chi \in \hat{G}$. A third characterization of this element is in term of partial zeta functions. For $\sigma \in G$, the partial zeta function associated to g (and the extension K/k and the set S) is defined, for $\mathrm{Re}(s) > 1$, by

$$\zeta_{K/k,S}(s,\sigma) := \sum_{\substack{(\mathfrak{a},S)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \mathcal{N}(\mathfrak{a})^{-1}$$

where $\mathfrak{a}$ runs through the integral ideals of k, not divisible by the prime ideals in S and whose Artin symbol $\sigma_{\mathfrak{a}} \in G$ is equal to σ. These functions also admit meromorphic continuation to the complex plan and are related to Hecke L-functions by the formula

$$L_{K/k,S}(s,\chi) = \sum_{\sigma \in G} \zeta_{K/k,S}(s,\sigma)\chi(\sigma).$$

From this we deduce the third characterization of the Brumer-Stickelberger element

$$\theta_{K/k,S} = \sum_{g \in G} \zeta_{K/k,S}(0,\sigma)\sigma^{-1}.$$

It follows from the Siegel-Klingen theorem that the values of the partial zeta functions at $s = 0$ are rational, thus $\theta_{K/k,S} \in \mathbb{Q}[G]$. A more precise result of Deligne et Ribet [5] (see also Barsky [1] and Cassou-Noguès [3]) states that, for any $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, the annihilator in $\mathbb{Z}[G]$ of the group μ_K of roots of unity in K, we have $\xi\, \theta_{K/k,S} \in \mathbb{Z}[G]$. In particular, if we let w_K denote the cardinality of μ_K, we have

$$w_K \theta_{K/k,S} \in \mathbb{Z}[G]. \tag{2.2}$$

We need one last notation before stating the abelian Brumer-Stark conjecture. We say that a non-zero element α in K is an anti-unit if all its conjugate have absolute value equal to 1. The group of anti-units of K is denoted by K°.

Conjecture 2.1 (Brumer-Stark conjecture $\mathbf{BS}(K/k,S)$)**.** *For any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{w_K\theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})/k$ is abelian.*

Remark. The last assertion that $K(\alpha^{1/w_K})/k$ is abelian does not depend on the choice of the w_K-th root of α since all these roots generate the same extension of K.

Let v be a place in S and denote by $N_v := \sum_{\sigma\in D_v} \sigma \in \mathbb{Z}[G]$ the sum of all the elements in the decomposition group D_v of v in G. Then, one can prove, see [14, Chap. IV], that

$$N_v\, \theta_{K/k,S} = 0. \tag{2.3}$$

In particular, if the set S contains a place that is totally split in K/k, the Brumer-Stickelberger element is equal to 0 and the abelian Brumer-Stark conjecture is trivially true. Therefore, the conjecture is only meaningful when k is not totally real and K totally complex.[1] In [13], Tate proves equivalent formulations of the conjecture that are very useful for its study. We will later on generalize this result to the non-abelian Galois case.

Theorem 2.2 (Tate)**.** *Let $\mathfrak{A}$ be a fractional ideal of K. Then the following statements are equivalent.*

- (i). *There exists an anti-unit $\alpha \in K^\circ$ such that $\mathfrak{A}^{w_K\theta_{K/k,S}} = \alpha\mathcal{O}_K$ and $K(\alpha^{1/w_K})/k$ is abelian.*
- (ii). *There exists an extension L/K such that L/k is abelian and an anti-unit $\gamma \in L^\circ$ such that $(\mathfrak{A}\mathcal{O}_L)^{\theta_{K/k,S}} = \gamma\mathcal{O}_L$.*
- (iii). *For almost all prime ideals[2] $\mathfrak{p}$ of k,there exists $\alpha_\mathfrak{p} \in K^\circ$ such that $\alpha_\mathfrak{p} \equiv 1 \pmod{^* \mathfrak{p}\mathcal{O}_K}$ and $\mathfrak{A}^{(\sigma_\mathfrak{p}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}} = \alpha_\mathfrak{p}\mathcal{O}_K$ where $\sigma_\mathfrak{p}$ is the Frobenius automorphism of $\mathfrak{p}$ in G.*
- (iv). *There exist a family $(a_i)_{i\in I}$ of element of $\mathbb{Z}[G]$ generating $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and a family $(\alpha_i)_{i\in I}$ of anti-units in K such that $\mathfrak{A}^{a_i\theta_{K/k,S}} = \alpha_i\mathcal{O}_K$ for all $i \in I$, and ${\alpha_i}^{a_j} = {\alpha_j}^{a_i}$ for all $i, j \in I$.*

Remark. In part (ii), $(\mathfrak{A}\mathcal{O}_L)^{\theta_{K/k,S}}$ is defined by the formula $\left((\mathfrak{A}\mathcal{O}_L)^{n\theta_{K/k,S}}\right)^{1/n}$ where $n \geq 1$ is any integer such that $n\theta_{K/k,S} \in \mathbb{Z}[G]$.

Let $\mathfrak{A}$ be a fractional ideal of K. We say that $\mathbf{BS}(K/k, S; \mathfrak{A})$ holds if the ideal $\mathfrak{A}$ satisfies the equivalent condition of the theorem. The conjecture $\mathbf{BS}(K/k, S)$ is thus the collection of the conjectures $\mathbf{BS}(K/k, S; \mathfrak{A})$ where $\mathfrak{A}$ ranges through the fractional ideals of K. In [13], Tate proves that the set of fractional ideals $\mathfrak{A}$ of K such that $\mathbf{BS}(K/k, S; \mathfrak{A})$ holds is a group, stable under the action of G, and containing the principal ideals of K. In particular, $\mathbf{BS}(K/k, S)$ holds if the field K is principal. Now, let $\mathfrak{p}_0$ be a prime ideal of k not in S, then

$$\theta_{K/k,S\cup\{\mathfrak{p}_0\}} = (1 - {\sigma_{\mathfrak{p}_0}}^{-1})\theta_{K/k,S}. \tag{2.4}$$

It follows from this formula that the validity of $\mathbf{BS}(K/k, S)$ implies that of $\mathbf{BS}(K/k, S \cup \{\mathfrak{p}_0\})$. Therefore, the conjecture is true for any admissible set of places S if it is true for the minimal choice of S formed exactly of the infinite places of k and of the finite places that ramify of K/k.

The validity of conjecture is also preserved under change of extension as a consequence of part (ii) of Proposition 2.2. That is, for $K/K'/k$ a tower of extension, the validity of $\mathbf{BS}(K/k, S)$ implies that of $\mathbf{BS}(K'/k, S)$. It also preserved under change of base, that is if $\mathbf{BS}(K/k, S)$ holds then does also $\mathbf{BS}(K/k', S')$ where $K/k'/k$ is a tower of extensions and S' denotes the set of places of k' above the place in k, see [8]. The following cases of the conjecture are proved by Tate (see [13] or [14]).

Theorem 2.3 (Tate)**.** *The abelian Brumer-Stark conjecture $\mathbf{BS}(K/k, S)$ is true in the following cases*

- *The field k is the field $\mathbb{Q}$ of rational numbers.*[3]
- *The extension K/k is quadratic.*

[1] Note that $K^\circ = \{\pm 1\}$ if K is not totally complex.

[2] Here and in the rest of the paper, when we say "for almost all prime ideals", we always implicitly exclude the ramified primes; therefore the Frobenius automorphism is always uniquely defined.

[3] In this situation, it boils down to Stickelberger theorem on cyclotomic sums.

- *The extension K/k is of degree 4 and contained into a non-abelian Galois extension K/k_0 of degree 8.*

Sands proves the abelian Brumer-Stark conjecture in certain cases when the group G has exponent 2. We refer the interested reader to [12] for more precise statements. In [7], a local version of the conjecture is stated and is proved in some cases and numerically studied in some other. The recent results of Greither and Popescu [6] implies that the local abelian Brumer-Stark conjecture at p holds provided that S contains all the primes above p and some appropriate Iwasawa μ-invariant vanishes.

3. The Galois Brumer-Stickelberger element

We assume from now on that the extension K/k is Galois, but not necessarily abelian. The set S still denotes a finite set of places of k, of cardinality at least 2, containing the infinite places of k and the finite places that ramify in K/k. The first step in our generalization of the abelian Brumer-Stark conjecture is to generalize the construction of the Brumer-Stickelberger element. Fortunately, such a construction is provided by the work of Hayes [9]. We now review his construction and the first properties of the Brumer-Stickelberger element. Denote by $\hat{G}$ the set of irreducible characters of G. For $\chi \in \hat{G}$, let $L_{K/k,S}(s,\chi)$ denote the Artin L-function of χ with Euler factors at primes in S deleted. The Brumer-Stickelberger element is defined by

$$\theta_{K/k,S} = \sum_{\chi \in \hat{G}} L_{K/k,S}(0,\chi)\, e_{\bar{\chi}} \tag{3.5}$$

where $e_\chi := \dfrac{\chi(1)}{|G|} \displaystyle\sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$ is the central idempotent of χ.

The following results are extracted from [9].

Theorem 3.1 (Hayes)**.** *Denote by $\mathscr{C}_G$ the set of conjugacy classes of G. The Brumer-Stickelberger element belongs to the center $Z(\mathbb{C}[G])$ of $\mathbb{C}[G]$ and is the only element of $\mathbb{C}[G]$ such that*

$$\phi_\chi(\theta_{K/k,S}) = L_{K/k,S}(0,\bar{\chi}) \tag{3.6}$$

for all $\chi \in \hat{G}$ where ϕ_χ is the ring homomorphism from $Z(\mathbb{C}[G])$ to $\mathbb{C}$ defined by

$$\phi_\chi(C) := \frac{\chi(C)}{\chi(1)}$$

for all $C \in \mathscr{C}_G$.

Let B be a normal subgroup of G. Then we have

$$\theta_{K^B/k,S} = \pi(\theta_{K/k,S})$$

where $\pi : \mathrm{Gal}(K/k) \to \mathrm{Gal}(K^B/k)$ is the canonical surjection induced by the restriction to K^B.

Let H be a subgroup of G. Denote by S_H the set of places of K^H above the places in S. Let $\mathrm{INorm}_{G\to H} : Z(\mathbb{C}[G]) \longrightarrow Z(\mathbb{C}[H])$ be the inhomogeneous norm defined by

$$\mathrm{INorm}_{G\to H}(a) := \sum_{\phi \in \hat{H}} \Big(\prod_{\chi \in \hat{G}} a(\chi)^{\langle \chi, \mathrm{Ind}_H^G \phi \rangle_G} \Big) e_\phi$$

where $a := \sum_{\chi \in \hat{G}} a(\chi) e_\chi \in Z(\mathbb{C}[G])$, $\langle \cdot,\cdot \rangle_G$ is the inner product on the characters of G and e_ϕ is the central idempotent of $\mathbb{C}[H]$ associated to ϕ. Then we have

$$\theta_{K/K^H,S_H} = \mathrm{INorm}_{G\to H}(\theta_{K/k,S}).$$

Remark. In the proposition, we identified the conjugacy class $C \in \mathscr{C}_G$ with the element $\sum_{g\in C} g$ of the group ring $\mathbb{C}[G]$.

We are now interested in generalizing properties (2.3) and (2.4). We start with (2.3).

Proposition 3.2. *For v a place of k, define*

$$N_v := \sum_{\sigma \in D_w} \frac{1}{|C_\sigma|} C_\sigma \in \mathbb{Q}[G]$$

where w is a place of K above v, D_w is the decomposition group of w in K/k and $C_\sigma \in \mathscr{C}_G$ is the conjugacy class of σ in G. Then, for any place v in S, we have

$$N_v\, \theta_{K/k,S} = 0.$$

Proof. Since N_v is in $Z(\mathbb{C}[G])$, it is enough, with the notations of Theorem 3.1, to prove that $\phi_\chi(N_v\, \theta_{K/k,S}) = \phi_\chi(N_v)\phi_\chi(\theta_{K/k,S}) = 0$ for all $\chi \in \hat{G}$. Let $\chi \in \hat{G}$ be such that $\phi_\chi(N_v) \neq 0$. By (3.6), we need to prove that the order $r(\bar{\chi}) = r(\chi)$ of vanishing at $s = 0$ of $L_{K/k,S}(s,\chi)$ is at least 1. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation with character χ. By [14, Prop. I.3.4], we have

$$r(\chi) = \sum_{v' \in S} \dim V^{D_{w'}} - \dim V^G \tag{3.7}$$

where w' is a place of K above v' and $D_{w'}$ denotes the decomposition group of w' in G. Assume first that χ is the trivial character. Then the above formula yields $r(\chi) = |S| - 1$ and the result follows from our hypothesis that S contains at least two places. Assume now that χ is non-trivial. We compute

$$\phi_\chi(N_v) = \sum_{\sigma \in D_w} \frac{1}{|C_\sigma|}\phi_\chi(C_\sigma) = \frac{1}{\chi(1)} \sum_{\sigma \in D_w} \chi(\sigma) = \frac{|D_w|}{\chi(1)} \langle \mathbf{1}_{D_w}, \chi_{|D_w} \rangle_{D_w}$$

where $\mathbf{1}_{D_w}$ is the trivial character of D_w and $\langle \cdot, \cdot \rangle_{D_w}$ is the inner product of the space of characters of D_w. By the above hypothesis, $\phi_\chi(N_v) \neq 0$ and thus the trivial character $\mathbf{1_{D_w}}$ appears in the decomposition of $\chi_{|D_w}$. Therefore the space V^{D_w} has dimension at least 1. On the other hand, $V^G = \{0\}$ since χ is irreducible. It follows that $r(\chi) \geq 1$ and the result is proved. □

Assume that there exists $v \in S$ that is totally split in K/k. Then $N_v = 1$ and the Brumer-Stickelberger element is trivial in this case. Therefore, as in the abelian case, the Brumer-Stickelberger element is always trivial when k is not totally real or K not totally complex. In fact, we can say more than that. Recall that a number field E is CM if it is a totally complex quadratic extension of a totally real field. If furthermore E is Galois over some totally real subfield F, then $\mathrm{Gal}(E/F)$ has a unique complex conjugation and we say that a character χ of $\mathrm{Gal}(E/F)$ is totally odd if all the eigenvalues of some associated representation at the complex conjugation are equal to -1.

Proposition 3.3. *Let $\chi \in \hat{G}$ be a character such that $\phi_\chi(\theta_{K/k,S}) \neq 0$. Then χ is the inflation of a totally odd character of a Galois CM sub-extension F/k of K/k.*

Proof. Let χ be such a character. Since S contains at least two elements, the character χ cannot be trivial. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of character χ. Denote by $F := K^{\mathrm{Ker}(\rho)}$ the subfield of K fixed by the kernel of ρ, by $\tilde{G}$ the Galois group of F/k, and by $\tilde{\rho} : \tilde{G} \to \mathrm{GL}(V)$ the faithful irreducible representation such that $\rho = \tilde{\rho} \circ \pi$ where $\pi : G \to \tilde{G}$ is the canonical surjection induced by the restriction to F. Denote by $\tilde{\chi}$ the character of $\tilde{\rho}$. By the properties of Artin L-functions, we have $r(\tilde{\chi}) = r(\chi) = 0$. As $\tilde{\chi}$ is irreducible, $V^{\tilde{G}} = \{0\}$ and thus by (3.7), we must have $V^{D_{\tilde{w}}} = \{0\}$ for all the places $\tilde{w}$ of F above the places in S. In particular, all the infinite places of F must be complex and, for $\tilde{w}$ a complex place of F, the complex conjugation $\tau_{\tilde{w}}$ acts as -1 on V. Since ρ is faithful, it follows that all the complex conjugations of F are equal to, say, $\tau \in \tilde{G}$. Therefore $\{1, \tau\}$ is a normal subgroup of $\tilde{G}$ and its fixed field is totally real. This proves that F is CM, $\tilde{\chi}$ totally odd and concludes the proof. □

Corollary 3.4. *If K/k does not contain a Galois CM sub-extension then $\theta_{K/k,S} = 0$.*

Proof. Assume that $\theta_{K/k,S} \neq 0$. Then, by Theorem 3.1 and the fact that $(\phi_\chi)_{\chi\in\hat{G}}$ is a basis of the dual of $Z(\mathbb{C}[G])$, see [9], we get that there exists an irreducible character χ of G such that $r(\chi) = 0$. This character comes from a Galois CM sub-extension by the proposition. $\square$

Corollary 3.5. *Let τ be a complex conjugation of G. Then $(\tau + 1) \cdot \theta_{K/k,S} = 0$.*

Proof. By the proposition, it is enough to prove that $(\tau + 1) \cdot e_\chi = 0$ for any character $\chi \in \hat{G}$ that is the inflation of a totally odd character $\tilde{\chi}$ of a Galois CM sub-extension. Since $\tilde{\chi}$ is totally odd, we have $\chi(g\tau) = -\chi(g)$ for all $g \in G$. Let R be a set of representatives of $G/\{1, \tau\}$. We now compute

$$\begin{aligned}(\tau + 1) \cdot e_\chi &= (\tau + 1) \cdot \frac{\chi(1)}{|G|} \sum_{\rho\in R} \Big(\chi(\rho)\rho^{-1} + \chi(\rho\tau)(\rho\tau)^{-1}\Big)\\ &= (\tau + 1) \cdot \frac{\chi(1)}{|G|} \sum_{\rho\in R} \Big(\chi(\rho)\rho^{-1} - \chi(\rho)\tau\rho^{-1}\Big)\\ &= (\tau + 1)(1 - \tau) \cdot \frac{\chi(1)}{|G|} \sum_{\rho\in R} \chi(\rho)\rho^{-1} = 0.\end{aligned}$$ $\square$

The following result generalizes (2.4) to the Galois case.

Proposition 3.6. *Let $\mathfrak{p}_0$ be a prime ideal of k not in S. Then*

$$\theta_{K/k,S\cup\{\mathfrak{p}_0\}} = \theta_{K/k,S} \sum_{\chi\in\hat{G}} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_0}))e_{\bar{\chi}}$$

where $\mathfrak{P}_0$ is a prime ideal of K above $\mathfrak{p}_0$, $\sigma_{\mathfrak{P}_0}$ is the Frobenius automorphism of $\mathfrak{P}_0$ in G, and, for $\chi \in \hat{G}$, ρ_χ denotes an irreducible representation of G with character χ.

Proof. With the notations of Theorem 3.1, it is enough to prove, for all $\psi \in \hat{G}$, that

$$\begin{aligned}\phi_\psi(\theta_{K/k,S\cup\{\mathfrak{p}_0\}}) &= \phi_\psi(\theta_{K/k,S})\, \phi_\psi\Big(\sum_{\chi\in\hat{G}} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_0}))e_{\bar{\chi}}\Big)\\ &= L_{K/k,S}(0, \bar{\psi}) \sum_{\chi\in\hat{G}} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_0}))\phi_\psi(e_{\bar{\chi}}).\end{aligned}$$

On the other hand, from the definition of Artin L-functions, we see that

$$\phi_\psi(\theta_{K/k,S\cup\{\mathfrak{p}_0\}}) = L_{K/k,S\cup\{\mathfrak{p}_0\}}(0, \bar{\psi}) = L_{K/k,S}(0, \bar{\psi})\, \det(1 - \rho_{\bar{\psi}}(\sigma_{\mathfrak{P}_0})).$$

The result follows from the fact that $\phi_\psi(e_{\bar{\chi}}) = 1$ if $\psi = \bar{\chi}$ and zero otherwise. $\square$

We now turn to the question of the rationality of the Brumer-Stickelberger element $\theta_{K/k,S}$ when G is non-abelian. In fact, we will see that it is a consequence of the principal rank zero Stark conjecture, proved by Tate [14], that $\theta_{K/k,S}$ lies in $\mathbb{Q}[G]$. It is worth noting that the proof of the principal rank zero Stark conjecture uses as a key ingredient the fact that the values at $s = 0$ of partial zeta functions are rational. The principal rank zero Stark conjecture states that, for any character χ of G, we have

$$L_{K/k,S}(0, \chi^\alpha) = L_{K/k,S}(0, \chi)^\alpha \text{ for all } \alpha \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C}) \tag{3.8}$$

where $\chi^\alpha := \alpha \circ \chi$. We write

$$\theta_{K/k,S} = \sum_{\chi\in\hat{G}} L_{K/k,S}(0, \chi)\frac{\bar{\chi}(1)}{|G|} \sum_{\sigma\in G} \chi(\sigma)\sigma = \sum_{\sigma\in G} x_\sigma\, \sigma$$

where

$$x_\sigma := \frac{1}{|G|} \sum_{\chi \in \hat{G}} \bar{\chi}(1)\chi(\sigma) L_{K/k,S}(0,\chi).$$

Let α be an automorphism of $\mathbb{C}$. We compute

$$\begin{aligned}\alpha(x_\sigma) &= \frac{1}{|G|} \sum_{\chi \in \hat{G}} \bar{\chi}^\alpha(1)\chi^\alpha(\sigma) L_{K/k,S}(0,\chi)^\alpha \\ &= \frac{1}{|G|} \sum_{\chi \in \hat{G}} \bar{\chi}^\alpha(1)\chi^\alpha(\sigma) L_{K/k,S}(0,\chi^\alpha) = x_\sigma\end{aligned}$$

since the map $\chi \mapsto \chi^\alpha$ is a bijection on the set $\hat{G}$. It follows that $x_\sigma \in \mathbb{Q}$ and we have proved the following result.

Theorem 3.7. *The Brumer-Stickelberger element $\theta_{K/k,S}$ lies in $\mathbb{Q}[G]$.* □

An interesting problem is to find a suitable denominator for the Brumer-Stickelberger element in the non-abelian case. In the abelian case, as we noted above, $w_K \theta_{K/k,S}$ is always integral. In the Galois case, however, one can see on examples that it is not the case anymore. Let $[G,G]$ be the commutator subgroup of G, that is the subgroup generated by the commutators $[g_1,g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ with $g_1, g_2 \in G$. Recall that $\mathscr{C}_G$ is the set of conjugacy classes of G. We make the following conjecture.

Conjecture 3.8 (Integrality conjecture)**.** *Define m_G to be the lcm of the cardinalities of the conjugacy classes in $\mathscr{C}_G$ and let s_G be the order of the commutator subgroup $[G,G]$ of G. Define d_G to be the lcm of m_G and s_G. Then, for almost all prime ideals $\mathfrak{P}$ of K, we have*

$$d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S} \in \mathbb{Z}[G]$$

where $\mathfrak{p}$ is the prime ideal of k below $\mathfrak{P}$ and $\sigma_{\mathfrak{P}}$ is the Frobenius automorphism of $\mathfrak{P}$ in G.

Note that we have $m_G = 1$ if and only if $s_G = 1$ if and only if G is abelian. Therefore the conjecture is satisfied when the extension K/k is abelian and is equivalent in that case to the statement before (2.2) using Lemma 3.9 below. It is also satisfied in the special case that we study in Section 7 and in all the computations that we have performed [4].

Let $G^{\mathrm{ab}} := G/[G,G]$ be the maximal abelian quotient of G and $K^{\mathrm{ab}} = K^{[G,G]}$ be the maximal sub-extension of K/k that is abelian over k; we have $\mathrm{Gal}(K^{\mathrm{ab}}/k) = G^{\mathrm{ab}}$. Denote by $\pi^{\mathrm{ab}} : G \to G^{\mathrm{ab}}$ the canonical surjection induced by the restriction to K^{ab} and by ν^{ab} the map from $\mathbb{Z}[G^{\mathrm{ab}}]$ to $\mathbb{Z}[G]$ defined, for $\tilde{g} \in G^{\mathrm{ab}}$, by

$$\nu^{\mathrm{ab}}(\tilde{g}) := \frac{1}{s_G} \sum_{\pi^{\mathrm{ab}}(g) = \tilde{g}} g \tag{3.9}$$

where the sum is over elements $g \in G$ whose image by π^{ab} is equal to $\tilde{g}$ and extended linearly. The characters of degree 1 of G are exactly the ones that are inflation of characters of G^{ab}. Let χ be such a character and let $\tilde{\chi}$ denote the character of G^{ab} such that $\chi = \tilde{\chi} \circ \pi^{\mathrm{ab}}$. One checks readily that $e_\chi = \nu^{\mathrm{ab}}(e_{\tilde{\chi}})$ where $e_{\tilde{\chi}}$ is the idempotent of $\mathbb{C}[G^{\mathrm{ab}}]$ associated to $\tilde{\chi}$. By the properties of Artin L-functions, we have

$$\begin{aligned}\sum_{\substack{\chi \in \hat{G} \\ \chi(1)=1}} L_{K/k,S}(0,\chi) e_{\bar{\chi}} &= \sum_{\tilde{\chi} \in \hat{G}^{\mathrm{ab}}} L_{K^{\mathrm{ab}}/k,S}(0,\tilde{\chi}) \nu^{\mathrm{ab}}(e_{\bar{\tilde{\chi}}}) \\ &= \nu^{\mathrm{ab}}\Big(\sum_{\tilde{\chi} \in \hat{G}^{\mathrm{ab}}} L_{K^{\mathrm{ab}}/k,S}(0,\tilde{\chi}) e_{\bar{\tilde{\chi}}} \Big) = \nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S}).\end{aligned}$$

We define
$$\theta^{(>1)}_{K/k,S} := \sum_{\substack{\chi\in\hat{G}\\ \chi(1)>1}} L_{K/k,S}(0,\chi)e_{\bar{\chi}}.$$

By the above computation, we have
$$\theta_{K/k,S} = \nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S}) + \theta^{(>1)}_{K/k,S}. \tag{3.10}$$

A direct computation shows that, for $\xi \in \mathbb{C}[G]$, we have $\xi\nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S}) = \nu^{\mathrm{ab}}(\tilde{\xi}\theta_{K^{\mathrm{ab}}/k,S})$ where $\tilde{\xi} := \pi^{\mathrm{ab}}(\xi)$. Therefore, it follows from the remark before (2.2) that, for all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have $s_G\,\xi\,\nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S}) \in \mathbb{Z}[G]$. The next result is proved in [14, Lemme IV.1.1] for abelian extensions. It is straightforward to extend the proof to Galois extensions.

Lemma 3.9. *Let $\mathcal{T}$ be a set of prime ideals containing all the unramified prime ideals of K that do not divide w_K except, possibly, a finite number. Then the annihilator $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ of μ_K in $\mathbb{Z}[G]$ is generated as a $\mathbb{Z}$-module by the elements $\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})$ where $\mathfrak{P}$ runs through the prime ideals in $\mathcal{T}$, and $\mathfrak{p}$ denotes the prime ideal of k below $\mathfrak{P}$. Furthermore, we have*
$$w_K = \gcd_{\substack{\mathfrak{P}\in\mathcal{T}\\ \sigma_{\mathfrak{P}}=1}} (1-\mathcal{N}(\mathfrak{p})).$$
□

From this, we deduce equivalent formulations of the Integrality Conjecture.

Proposition 3.10. *The following assertions are equivalent*
(1). *For almost all prime ideals $\mathfrak{P}$ of K, $d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S} \in \mathbb{Z}[G]$;*
(2). *For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, $d_G\,\xi\,\theta_{K/k,S} \in \mathbb{Z}[G]$;*
(3). *For almost all prime ideals $\mathfrak{P}$ of K, $d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$;*
(4). *For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, $d_G\,\xi\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$.*

Proof. The equivalences (1) ⇔ (3) and (2) ⇔ (4) are consequences of the above discussion. The direction (2) ⇒ (1) is trivial. The other direction follows from the above lemma. □

4. Strong central extensions

Before we generalize the abelian Brumer-Stark conjecture to Galois extensions, we introduce the notion of strong central extensions that will play a primordial part. For that, we stop assuming for a moment that G is the Galois group of the extension K/k and just consider G as a group. Let Γ and Δ be two other groups with Δ a normal subgroup of Γ such that the following sequence is exact
$$1 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1, \tag{4.11}$$
that is, Γ is a group extension of G by Δ. We say that Γ is a strong central extension of G by Δ if $\Delta \cap [\Gamma,\Gamma] = 1$ where $[\Gamma,\Gamma]$ is the commutator subgroup of Γ. The choice of terminology is explained by the following lemma.

Lemma 4.1. *Let Γ be a strong central extension of G by Δ. Then Γ is a central extension of G by Δ.*

Proof. Let $\gamma \in \Gamma$ and $\delta \in \Delta$. We compute
$$\pi([\gamma,\delta]) = \pi(\gamma)\pi(\delta)\pi(\gamma)^{-1}\pi(\delta)^{-1} = \pi(\gamma)\pi(\gamma)^{-1} = 1.$$
Thus, $[\gamma, b] \in \Delta \cap [\Gamma,\Gamma] = \{1\}$ and γ and δ commute. Therefore Δ is in the center of Γ and the extension is central. □

The following two lemmas provide us with equivalent characterizations of strong central extensions.

Lemma 4.2. *Consider the group extension* (4.11)*. This extension is strong central if and only if, for any abelian subgroup H of G, the subgroup $\pi^{-1}(H)$ of Γ is abelian.*

Proof. Assume that the extension is strong central. Let H be an abelian subgroup of G. Let $\gamma_1, \gamma_2 \in \pi^{-1}(H)$, say $\pi(\gamma_1) = h_1$, $\pi(\gamma_2) = h_2$ with $h_1, h_2 \in H$. We compute

$$\pi([\gamma_1, \gamma_2]) = [h_1, h_2] = 1.$$

By hypothesis, this implies that $[\gamma_1, \gamma_2] = 1$ and therefore $\pi^{-1}(H)$ is abelian.

Reciprocally, we assume that, for any abelian subgroup H of G, the group $\pi^{-1}(H)$ is abelian. Let $\gamma_1, \gamma_2 \in \Gamma$ be such that $[\gamma_1, \gamma_2] \in \Delta$. Then $\pi([\gamma_1, \gamma_2]) = 1$ and $\pi(\gamma_1)$ and $\pi(\gamma_2)$ commute. The subgroup of G that they generate is abelian and, by hypothesis, it follows that γ_1 and γ_2 commute, that is $[\gamma_1, \gamma_2] = 1$. Therefore the extension Γ of G by Δ is strong central. □

Lemma 4.3. *Consider the group extension* (4.11)*. This extension is strong central if and only if the map s restricts to an isomorphism between $[\Gamma, \Gamma]$ and $[G, G]$.*

Proof. It is direct to see that π restricts to a surjective map from $[\Gamma, \Gamma]$ to $[G, G]$. This map is injective if and only if $[\Gamma, \Gamma] \cap \mathrm{Ker}(\pi) = 1$. The result follows since $\mathrm{Ker}(\pi) = \Delta$. □

We note another property of strong central extensions that will be useful later on. For a finite group A, recall that m_A denote the lcm of the cardinalities of the conjugacy classes of A, s_A is the order of the commutator subgroup $[A, A]$ of A and d_A is the lcm of m_A and s_A.

Lemma 4.4. *Consider the group extension* (4.11)*. Assume that the extension is strong central and that Γ is finite. Then we have $d_\Gamma = d_G$.*

Proof. It is enough to show that $m_\Gamma = m_G$ and $s_\Gamma = s_G$. The fact that $s_\Gamma = s_G$ is a direct consequence of the previous lemma. We now show that $m_\Gamma = m_G$. Let $\gamma \in \Gamma$. Denote by C and Z respectively the conjugacy class of γ in Γ and the centralizer of γ in Γ. We have

$$\begin{aligned} |C| &= (\Gamma : Z) = (\pi(\Gamma) : \pi(Z))(\mathrm{Ker}(\pi) : \mathrm{Ker}(\pi) \cap Z) = (G : \pi(Z))(\Delta : \Delta \cap Z) \\ &= (G : Z_0)(Z_0 : \pi(Z))(\Delta : \Delta \cap Z) = |C_0|(Z_0 : \pi(Z))(\Delta : \Delta \cap Z) \end{aligned}$$

where C_0 is the conjugacy class of $\pi(\gamma)$ in G and Z_0 is the centralizer of $\pi(\gamma)$ in G. Since Δ is in the center of Γ by Lemma 4.1, we have $\Delta \subset Z$ and $(\Delta : \Delta \cap Z) = 1$. Now, let $\rho_0 \in Z_0$ and let $\rho \in \pi^{-1}(\rho_0)$. We have $\pi([\rho, \gamma]) = [\rho_0, \pi(\gamma)] = 1$ since ρ_0 commutes with $\pi(\gamma)$. Therefore $[\rho, \gamma] \in [\Gamma, \Gamma] \cap \Delta = \{1\}$ and $\rho \in Z$. Thus, $\pi(Z) = Z_0$ and we have finally $|C| = |C_0|$. As any conjugacy class of G is the image by π of a conjugacy class of Γ, we have $m_\Gamma = m_G$ and the result is proved. □

We now come back to our previous setting and assume that G is the Galois group of the extension K/k. Let L be an extension of K. We say that L is a strong central extension of K/k if L/k is Galois and the group extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

is strong central where $\Delta := \mathrm{Gal}(L/K)$ and $\Gamma := \mathrm{Gal}(L/k)$. The following result is a direct consequence of the definition of strong central extensions.

Lemma 4.5. *Denote by L^{ab} the maximal sub-extension of L/k that is abelian over k. Then L is a strong central extension of K/k if and only if $L = KL^{\mathrm{ab}}$. Furthermore, in that case, restriction to L^{ab} yields an isomorphism between $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$ where K^{ab} is the maximal sub-extension of K/k that is abelian over k.* □

We conclude this section with a lemma that shows central extensions behave somewhat nicely.

Lemma 4.6. *Let L be a strong central extension of K/k.*

(1) Let L_0/K be a sub-extension of L/K. Then L_0 is a strong central extension of K/k.

(2) *Let M be another strong central extensions of K/k. Then LM is a strong central extension of K/k.*

Proof. We prove the first assertion. The group $\mathrm{Gal}(L/L_0)$ is a subgroup of $\mathrm{Gal}(L/K)$ and thus it is normal in $\mathrm{Gal}(L/k)$. Therefore, L_0/k is a Galois extension. Let $L_0^{\mathrm{ab}} = L^{\mathrm{ab}} \cap L_0$ be the maximal abelian sub-extension of L_0/k. It follows from the above lemma that $[L_0^{\mathrm{ab}} : K^{\mathrm{ab}}] = [L_0 : K]$. Since $L_0^{\mathrm{ab}} \cap K = K^{\mathrm{ab}}$, we find that

$$[KL_0^{\mathrm{ab}} : k] = \frac{[L_0^{\mathrm{ab}} : k][K : k]}{[K^{\mathrm{ab}} : k]} = [L_0^{\mathrm{ab}} : K^{\mathrm{ab}}][K : k] = [L_0 : k],$$

thus $KL_0^{\mathrm{ab}} = L_0$ and L_0 is a strong central extension of K/k by the previous lemma.

We now prove the second assertion. The extension LM/k is Galois as the compositum of two Galois extensions of k. Let $F = L \cap M$. It is an extension of K. Then, a direct computation shows that $[LM : K] = [L^{\mathrm{ab}}M^{\mathrm{ab}} : K^{\mathrm{ab}}]$. We find that

$$[KL^{\mathrm{ab}}M^{\mathrm{ab}} : k] = \frac{[L^{\mathrm{ab}}M^{\mathrm{ab}} : k][K : k]}{[K^{\mathrm{ab}} : k]} = [L^{\mathrm{ab}}M^{\mathrm{ab}} : K^{\mathrm{ab}}][K : k] = [LM : k].$$

Thus, $KL^{\mathrm{ab}}M^{\mathrm{ab}} = LM$. Since the maximal abelian sub-extension $(LM)^{\mathrm{ab}}$ of LM/k that is abelian over k contains $L^{\mathrm{ab}}M^{\mathrm{ab}}$, it follows that $K(LM)^{\mathrm{ab}} = LM$ and LM is a strong central extension of K/k. ☐

5. The Galois Brumer-Stark conjecture

We now generalize the abelian Brumer-Stark conjecture to Galois extensions.

Conjecture 5.1 ($\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$)**.** *The Integrality Conjecture holds and, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G w_K \theta_{K/k,S}}$ is principal, and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})$ is a strong central extension of K/k.*

Remark. As in the abelian case, the last assertion that $K(\alpha^{1/w_K})$ is a strong central extension of K/k does not depend on the choice of the w_K-th root of α since all of these generate the same extension of K.

Before studying the conjecture, we discuss briefly our evidence for it. The first evidence is that the conjecture is in many ways a natural generalization of the abelian Brumer-Stark conjecture. Indeed, it is equivalent to it in the abelian case, see below, and share many properties similar to it, see next section. We prove also in the last section that, in one special setting, the Galois Brumer-Stark conjecture is implied by the abelian Brumer-Stark conjecture. Finally, in a forthcoming paper [4], we prove numerically that the conjecture holds in many examples.

Proposition 5.2. *Assume that K/k is abelian. Then the Galois Brumer-Stark conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ is equivalent to the abelian Brumer-Stark conjecture $\mathbf{BS}(K/k, S)$.*

Proof. This is clear since $d_G = 1$ in that case and, by Lemma 4.2, we see that $K(\alpha^{1/w_K})/k$ is abelian if and if only if $K(\alpha^{1/w_K})$ is as strong central extension of K/k. ☐

Assume that the Integrality Conjecture holds. For a fractional ideal $\mathfrak{A}$ of K, we say that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ is satisfied if the ideal $\mathfrak{A}$ verifies the properties stated in the conjecture. The conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ is thus equivalent to the Integrality Conjecture and the collection of the conjectures $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ where $\mathfrak{A}$ ranges through the fractional ideals of K. The following result gives equivalent formulations for $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ and is the generalization of Theorem 2.2. Recall that, for a prime ideal $\mathfrak{P}$ of K, we denote by $\mathfrak{p}$ the prime ideal of k below $\mathfrak{P}$ and by $\sigma_{\mathfrak{P}}$ the Frobenius automorphism of $\mathfrak{P}$ in G.

Theorem 5.3. *Assume the Integrality Conjecture holds. Let $\mathfrak{A}$ be a fractional ideal of K. The following assertions are equivalent.*

(i). $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ *is satisfied, that is there exists an anti-unit* $\alpha \in K^\circ$ *such that* $\mathfrak{A}^{d_G w_K \theta_{K/k,S}} = \alpha\mathcal{O}_K$ *and* $K(\alpha^{1/w_K})$ *is a strong central extension of* K/k.

(ii). *There exists an extension* L/K *that is a strong central extension of* K/k *and an anti-unit* $\gamma \in L^\circ$ *such that* $(\mathfrak{A}\mathcal{O}_L)^{d_G\theta_{K/k,S}} = \gamma\mathcal{O}_L$

(iii). *For almost all prime ideals* $\mathfrak{P}$ *of* K, *there exists an anti-unit* $\alpha_{\mathfrak{P}} \in K^\circ$ *such that* $\mathfrak{A}^{d_G(\sigma_{\mathfrak{P}}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}} = \alpha_{\mathfrak{P}}\mathcal{O}_K$ *and* $\alpha_{\mathfrak{P}} \equiv 1 \pmod{^* \mathfrak{Q}}$ *for all prime ideals* $\mathfrak{Q}$ *of* K *above* $\mathfrak{p}$ *such that* $\sigma_{\mathfrak{Q}} = \sigma_{\mathfrak{P}}$.

(iv). *For any abelian subgroup* H *of* G, *there exists a finite family* $(a_i)_{i\in I}$ *of elements of* $\mathbb{Z}[H]$ *generating* $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$ *as a* $\mathbb{Z}$*-module and anti-units* $(\alpha_i)_{i\in I}$ *of* K *such that* $\mathfrak{A}^{d_G a_i\theta_{K/k,S}} = \alpha_i\mathcal{O}_K$ *and* ${\alpha_j}^{a_i} = {\alpha_i}^{a_j}$ *for all* $i,j \in I$.

Remark. In part *(ii)*, $(\mathfrak{A}\mathcal{O}_L)^{d_G\theta_{K/k,S}}$ is defined by the formula $\big((\mathfrak{A}\mathcal{O}_L)^{nd_G\theta_{K/k,S}}\big)^{1/n}$ where $n \geq 1$ is any integer such that $nd_G\theta_{K/k,S} \in \mathbb{Z}[G]$.

Proof. We use repeatedly the fact that $\theta_{K/k,S}$ lies in the center of $\mathbb{C}[G]$.

(i) $\Rightarrow$ *(ii)*. Let $\gamma := \alpha^{1/w_K}$ and $L := K(\gamma)$. Then, L is a strong central extension of K/k and γ is an anti-unit in L. Furthermore, we have

$$\big(\gamma\mathcal{O}_L\big)^{w_K} = \alpha\mathcal{O}_L = (\mathfrak{A}\mathcal{O}_L)^{d_G w_K\theta_{K/k,S}}$$

and the result follows since the group of ideals of a number field is torsion-free.

(ii) $\Rightarrow$ *(iii)*. Denote by Γ the Galois group of L/k and by Δ the Galois group of L/K. Let $\mathcal{T}$ be the set of prime ideals of K, unramified in L/K and $K/\mathbb{Q}$, relatively prime with w_K and with $\mathfrak{A}$ and all its conjugates over k. Note that $\mathcal{T}$ contains all but finitely many prime ideals of K. Let $\mathfrak{P} \in \mathcal{T}$ and let $\tilde{\mathfrak{P}}$ be a prime ideal of L above $\mathfrak{P}$. Denote by $\sigma_{\tilde{\mathfrak{P}}}$ the Frobenius automorphism of $\tilde{\mathfrak{P}}$ in Γ. We set $\alpha_{\tilde{\mathfrak{P}}} := \gamma^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})}$. Let $\tilde{\mathfrak{Q}}$ be another prime ideal of L above $\mathfrak{p}$ such that $\pi(\sigma_{\tilde{\mathfrak{P}}}) = \pi(\sigma_{\tilde{\mathfrak{Q}}})$ where $\pi : \Gamma \to G$ is the canonical surjection induced by the restriction to K and $\sigma_{\tilde{\mathfrak{Q}}}$ is the Frobenius automorphism of $\tilde{\mathfrak{Q}}$ in Γ. Then, there exists $\rho \in \Gamma$ such that $\tilde{\mathfrak{Q}} = \rho(\tilde{\mathfrak{P}})$, and we have $\sigma_{\tilde{\mathfrak{Q}}} = \rho\sigma_{\tilde{\mathfrak{P}}}\rho^{-1}$. Since $\pi([\rho,\sigma_{\tilde{\mathfrak{P}}}]) = \pi(\sigma_{\tilde{\mathfrak{Q}}})\pi(\sigma_{\tilde{\mathfrak{P}}})^{-1} = 1$, it lies in Δ and is therefore trivial. Thus $\sigma_{\tilde{\mathfrak{Q}}} = \sigma_{\tilde{\mathfrak{P}}}$ and $\alpha_{\tilde{\mathfrak{Q}}} = \alpha_{\tilde{\mathfrak{P}}}$. In particular, $\alpha_{\tilde{\mathfrak{P}}}$ does not depend on the choice of the prime ideal $\tilde{\mathfrak{P}}$ of L above $\mathfrak{P}$, and we can just denote it by $\alpha_{\mathfrak{P}}$. Furthermore, $\alpha_{\mathfrak{P}} = \gamma^{\sigma_{\tilde{\mathfrak{Q}}}-\mathcal{N}(\mathfrak{p})} \equiv 1 \pmod{^* \tilde{\mathfrak{Q}}}$ for all prime ideals $\tilde{\mathfrak{Q}}$ of L above $\mathfrak{p}$ such that $\sigma_{\mathfrak{Q}} = \sigma_{\mathfrak{P}}$ where $\mathfrak{Q}$ is the prime ideal of K below $\tilde{\mathfrak{Q}}$. We now prove that $\alpha_{\mathfrak{P}}$ lies in K. Let $\rho \in \mathrm{Gal}(L/K)$. We have

$$\big(\alpha_{\mathfrak{P}}^{\rho-1}\big)^{w_K} = \Big(\big(\gamma^{w_K}\big)^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})}\Big)^{\rho-1} = \Big(\alpha^{\sigma_{\mathfrak{P}}-\mathcal{N}(\mathfrak{p})}\Big)^{\rho-1} = 1$$

since α lies in K. Thus, there exists a root of unity $\xi \in \mu_K$ such that $\alpha_{\mathfrak{P}}^{\rho-1} = \xi$. We have $\alpha_{\mathfrak{P}} \equiv \alpha_{\mathfrak{P}}^{\rho} \equiv 1 \pmod{^* \tilde{\mathfrak{P}}}$ by the above remark, hence $\xi \equiv 1 \pmod{^* \tilde{\mathfrak{P}}}$ and thus $\xi = 1$ by the choice of $\mathfrak{P}$. Therefore, $\alpha_{\mathfrak{P}} \in K$ as desired. Furthermore, it is clear from its construction that it is an anti-unit and we have $\alpha_{\mathfrak{P}} \equiv 1 \pmod{^* \mathfrak{Q}}$ for all prime ideals $\mathfrak{Q}$ above $\mathfrak{p}$ such that $\sigma_{\mathfrak{Q}} = \sigma_{\mathfrak{P}}$ by the above. We have

$$\alpha_{\mathfrak{P}}\mathcal{O}_L = (\gamma\mathcal{O}_L)^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})} = \big((\mathfrak{A}\mathcal{O}_L)^{d_G\theta_{K/k,S}}\big)^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})} = (\mathfrak{A}\mathcal{O}_L)^{d_G(\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}},$$

and, since $\mathfrak{A}$ is an ideal of K and $d_G(\sigma_{\mathfrak{P}}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S} \in \mathbb{Z}[G]$ by the Integrality Conjecture, we get

$$\alpha_{\mathfrak{P}}\mathcal{O}_K = \mathfrak{A}^{d_G(\sigma_{\mathfrak{P}}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}}.$$

The implication is proved.

(iii) $\Rightarrow$ *(iv)*. Let H be an abelian subgroup of G. Denote by $\mathcal{T}_H$ the subset of prime ideals of K for which *(iii)* applies, that are unramified in L/K and K/k, relatively prime with w_K and with $\mathfrak{A}$ and all its conjugates over k, and whose Frobenius automorphism in G actually lies in H. Let I be a set indexing $\mathcal{T}_H$, so that $\mathcal{T}_H = \{\mathfrak{P}_i : i \in I\}$. For $i \in I$, we set $a_i := \sigma_{\mathfrak{P}_i} - \mathcal{N}(\mathfrak{p}_i) \in \mathbb{Z}[H]$ and $\alpha_i := \alpha_{\mathfrak{P}_i} \in K^\circ$. It follows from Lemma 3.9 that the family $(a_i)_{i\in I}$ generates $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$.

By construction, we have also $\mathfrak{A}^{d_G a_i \theta_{K/k,S}} = \alpha_i \mathcal{O}_K$. It remains to prove that, for $i, j \in I$, we have $\alpha_j^{a_i} = \alpha_i^{a_j}$, that is, for two prime ideas $\mathfrak{P}$ and $\mathfrak{Q}$ in $\mathcal{T}_H$, the two elements $\alpha_{\mathfrak{P}}^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{q})}$ and $\alpha_{\mathfrak{Q}}^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})}$ are equal. We have

$$\begin{aligned}(\alpha_{\mathfrak{P}} \mathcal{O}_K)^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{p})} &= \left(\mathfrak{A}^{d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S}}\right)^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{p})} \\ &= \left(\mathfrak{A}^{d_G(\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S}}\right)^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})} = (\alpha_{\mathfrak{Q}} \mathcal{O}_K)^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})}\end{aligned}$$

where we used the fact that $\sigma_{\mathfrak{P}}$ and $\sigma_{\mathfrak{Q}}$ commute since they both belong to H. Since $\alpha_{\mathfrak{P}}$ and $\alpha_{\mathfrak{Q}}$ are both anti-units, there exists a root of unity $\xi \in \mu_K$ such that $\alpha_{\mathfrak{P}}^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{q})} = \xi \alpha_{\mathfrak{Q}}^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})}$. Reasoning as above, we see that $\xi \equiv 1 \pmod{^* \mathfrak{P}}$, thus $\xi = 1$ and the equality is proved.

$(iv) \Rightarrow (i)$. Let H be an abelian subgroup of G. Let $(a_i)_{i \in I}$ and $(\alpha_i)_{i \in I}$ be the corresponding families. There exists a family $(\lambda_i)_{i \in I}$ of integers, with only finitely many non-zero terms, such that

$$w_K = \sum_{i \in I} \lambda_i a_i.$$

In the same way, for any $h \in H$, there exists an integer $n_h \in \mathbb{N}$ such that $h - n_h$ annihilates μ_K. Therefore, there exists a family $(\lambda_{h,i})_{i \in I}$ of integers, with only finitely many non-zero terms, such that

$$h - n_h = \sum_{i \in I} \lambda_{h,i} a_i.$$

We set $\alpha_H := \prod_{i \in I} \alpha_i^{\lambda_i}$. It is clear that α_H is an anti-unit of K and we have

$$\alpha_H \mathcal{O}_K = \mathfrak{A}^{d_G(\sum_i \lambda_i a_i)\theta} = \mathfrak{A}^{d_G w_K \theta_{K/k,S}}.$$

In particular, up to a root of unity in K, α_H does not depend on the choices made, and we will therefore denote it simply by α. Furthermore, for $h \in H$, we have

$$\alpha^{h - n_h} = \prod_{i \in I} \big(\prod_{j \in I} \alpha_i^{a_j \lambda_{h,j}} \big)^{\lambda_i} = \prod_{i \in I} \big(\prod_{j \in I} \alpha_j^{\lambda_{h,j}} \big)^{\lambda_i a_i} = \alpha_h^{\sum_{i \in I} \lambda_i a_i} = \alpha_h^{w_K}$$

where $\alpha_h := \prod_{i \in I} \alpha_i^{\lambda_{h,i}}$. For g, another element of H, one can prove in the same way that $\alpha_h^{g - n_g} = \alpha_g^{h - n_h}$. Let $\gamma := \alpha^{1/w_K}$ and $L := K(\gamma)$. We now prove that L/K^H is an abelian extension. First, we prove that L/K^H is a Galois extension. For $h \in H$, let $\tilde{h}$ be any lift of h to L. We compute

$$(\gamma^{\tilde{h} - n_h})^{w_K} = (\gamma^{w_K})^{\tilde{h} - n_h} = \alpha^{h - n_h} = \alpha_h^{w_K}.$$

Thus, there exists $\xi_h \in \mu_K$ such that $\gamma^{\tilde{h} - n_h} = \xi_h \alpha_h$. Therefore, we have

$$\gamma^{\tilde{h}} = \xi_h \alpha_h \gamma^{n_h} \in L$$

and L/K^H is a Galois extension. We now prove that $\mathrm{Gal}(L/K^H)$ is abelian. Let $\tilde{h}, \tilde{g}$ be two elements of $\mathrm{Gal}(L/K^H)$; denote by h and g their restriction to K. We have

$$\gamma^{(\tilde{g} - n_g)(\tilde{h} - n_h)} = (\xi_h \alpha_h)^{g - n_g} = \alpha_h^{g - n_g} = \alpha_g^{h - n_h} = (\xi_g \alpha_g)^{h - n_h} = \gamma^{(\tilde{h} - n_h)(\tilde{g} - n_g)}$$

and therefore $\gamma^{\tilde{g}\tilde{h}} = \gamma^{\tilde{h}\tilde{g}}$. Thus $\mathrm{Gal}(L/K^H)$ is abelian as desired. Since this is true for any abelian subgroup H of G, we get by Lemma 4.2 that L is a strong central extension of K/k. This concludes the proof. $\square$

6. Some properties of the Galois Brumer-Stark conjecture

In this section, we look at the properties satisfied by the Galois Brumer-Stark conjecture with in view the generalization of the properties of the abelian Brumer-Stark conjecture listed in Section 2. We start by proving that the set of fractional ideals that satisfy the Galois Brumer-Stark conjecture have properties similar to the abelian case.

Proposition 6.1. *The set of fractional ideals $\mathfrak{A}$ of K that satisfy $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ is a group, stable under the action of G and that contains the principal ideals of K.*

Proof. We first prove that this set is a group. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two fractional ideals of K such that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ and $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{B})$ hold. Let α and β the anti-units satisfying part (i) of Theorem 5.3 for the ideals $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then $\alpha\beta$ is an anti-unit such that $\alpha\beta\mathcal{O}_K = (\mathfrak{A}\mathfrak{B})^{d_G w_K \theta_{K/k,S}}$. Furthermore, since $K((\alpha\beta)^{1/w_K}) \subset K(\alpha^{1/w_K}, \beta^{1/w_K})$, it is a strong central extension of K/k by Lemma 4.6 and therefore $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A}\mathfrak{B})$ is satisfied. Thus the set of ideals $\mathfrak{A}$ such that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ is satisfied is a subgroup of the group of fractional ideals of K.

Let σ be an element of G. We now prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A}^\sigma)$ is satisfied if $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ holds. Since $\theta_{K/k,S}$ is in the center of $\mathbb{C}[G]$, α^σ is a generator of

$$\left(\mathfrak{A}^{d_G w_K \theta_{K/k,S}}\right)^\sigma = \left(\mathfrak{A}^\sigma\right)^{d_G w_K \theta_{K/k,S}}.$$

Furthermore, α^σ is clearly an anti-unit. Let $\gamma := \alpha^{1/w_K}$ and $\delta := (\alpha^\sigma)^{1/w_K}$. Denote by $\tilde{\sigma}$ a lift of σ to $L := K(\gamma)$. Then there exists $\xi \in \mu_K$ such that $\delta = \xi\gamma^{\tilde{\sigma}}$. Since L/k is Galois, we get that $L' := K(\delta) \subset L$. This proves that L' is a strong central extension of K/k by Lemma 4.6 and thus concludes the proof that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A}^\sigma)$ is satisfied.

Finally, we prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S; \mathfrak{A})$ is satisfied if $\mathfrak{A}$ is a principal ideal, say $\mathfrak{A} = \eta\mathcal{O}_K$. For that, we use the equivalent formulation (iv) of Theorem 5.3. Let H be an abelian subgroup of G. For $h \in H$, let $n_h \in \mathbb{N}$ be such that $\xi^h = \xi^{n_h}$ for all $\xi \in \mu_K$ with the convention that $n_1 = w_K + 1$. Then the family $a_h := h - n_h$, for $h \in H$, generates $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$. For $h \in H$, we define $\alpha_h := \eta^{d_G a_h \theta_{K/k,S}}$. Note that $d_G a_h \theta_{K/k,S} \in \mathbb{Z}[G]$ by the Integrality Conjecutre. For all $h \in H$, we have $(\eta\mathcal{O}_K)^{d_G a_h \theta_{K/k,S}} = a_h\mathcal{O}_K$ by construction. Furthermore, let w be an infinite (complex) place of K. Denote by $\tau_w \in G$ the complex conjugation at w. By Corollary 3.5, we have that $(1 + \tau_w)\theta_{K/k,S} = 0$ and thus $\alpha_h^{1+\tau_w} = 1$ for all complex places w of K. Therefore α_h is an anti-unit for all $h \in H$. It remains to prove that $\alpha_h^{a_g} = \alpha_g^{a_h}$ for all $g, h \in H$. But this is a direct consequence of the fact that $(h - n_h)(g - n_g) = (g - n_g)(h - n_h)$ since H is abelian. This concludes the proof. $\square$

Corollary 6.2. *Assume that K is principal. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ is satisfied.* $\square$

Using the decomposition of the Brumer-Stickelberger element given by (3.10), we can prove the following result that relates $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ and $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$.

Proposition 6.3. *Assume that $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ holds. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ is satisfied if, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G w_K \theta^{(>1)}_{K/k,S}}$ is principal, and admits a generator $\beta \in K^\circ$ such that $K(\beta^{1/w_K})$ is a strong central extension of K/k.*

Proof. Let $\mathfrak{A}$ be a fractional ideal of K. Set $\mathfrak{a} := N_{K/K^{\mathrm{ab}}}(\mathfrak{A})$. An direct computation shows that

$$\mathfrak{A}^{d_G w_K \nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S})} = \mathfrak{a}^{(d_G/s_G) w_K \theta_{K^{\mathrm{ab}}/k,S}}\mathcal{O}_K.$$

By hypothesis, there exists α_0, an anti-unit in K^{ab}, such that $\mathfrak{a}^{(d_G/s_G) w_K \theta_{K^{\mathrm{ab}}/k,S}} = \alpha_0\mathcal{O}_{K^{\mathrm{ab}}}$ and $K^{\mathrm{ab}}(\alpha_0^{1/w_K})/k$ is abelian. Let $\alpha := \alpha_0\beta$. Then α is an anti-unit of K and by (3.10), we have

$$\alpha\mathcal{O}_K = \mathfrak{A}^{d_G w_K \theta_{K/k,S}}.$$

It remains to prove that $K(\alpha^{1/w_K})$ is a strong central extension of K/k. It is a sub-extension of $K(\alpha_0^{1/w_K}, \beta^{1/w_K})/K$. But $K(\beta^{1/w_K})$ is a strong central extension of K/k by hypothesis and $K(\alpha_0^{1/w_K})$ is a strong central extension of K/k by Lemma 4.5. Thus, $K(\alpha^{1/w_K})$ is a strong central extension of K/k by Lemma 4.6 and the result is proved. ☐

For $\chi \in \hat{G}$, recall that K^χ denote the subfield of K fixed by the kernel of χ.

Corollary 6.4. *Assume that* $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ *is satisfied and that, for all* $\chi \in \hat{G}$ *such that* $\chi(1) > 1$, K^χ *is not a CM extension. Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *holds.*

Proof. Indeed, in that case, $\theta_{K/k,S}^{(>1)} = 0$ by Proposition 3.3. ☐

We now turn to the question of the change of extension for the Galois Brumer-Stark conjecture. We will prove that it is satisfied in many cases up to a factor.

Proposition 6.5. *Let* K'/k *be a Galois sub-extension of* K/k *with* $G' := \mathrm{Gal}(K'/k)$. *Denote by* $\tilde{\mathbf{BS}}_{\mathrm{Gal}}(K'/k, S)$ *the Galois Brumer-Stark conjecture for the extension* K'/k *and the set of places* S *with the factor* $d_{G'}$ *replaced*[4] *by* d_G. *Assume that* w_K *is relatively prime with the degree of the extension* $K/K'K^{\mathrm{ab}}$. *Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *implies* $\tilde{\mathbf{BS}}_{\mathrm{Gal}}(K'/k, S)$.

Remark. If G is abelian then $K^{\mathrm{ab}} = K$, thus $K = K'K^{\mathrm{ab}}$ and the condition of the proposition is always satisfied. Furthermore, we have $d_G = d_{G'} = 1$ and we recover the fact that $\mathbf{BS}(K/k, S)$ implies $\mathbf{BS}(K'/k, S)$.

Remark. We prove actually a slighter stronger statement: if $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds then, for all fractional ideal $\mathfrak{A}'$ of K', there exists an anti-unit $\alpha \in K'$ such that

$$\mathfrak{A}'^{\,d_G w_{K'} \theta_{K'/k,S}} = (\alpha).$$

The extra hypothesis that w_K is relatively prime with the degree of $K/K'K^{\mathrm{ab}}$ is only used to prove that $K'(\alpha^{1/w_{K'}})$ is a strong central extension of K'/k.

In order to see that the statement of the proposition makes sense, we have the following lemma.

Lemma 6.6. *Let* A *be a finite group and let* B *be a quotient group of* A. *Then* d_B *divides* d_A.

Proof. It is enough to prove that s_B divides s_A and m_B divides m_A. Let $\pi : A \to B$ be the canonical surjection and denote by D its kernel. It is clear that s_B divides s_A since $\pi([A, A]) = [B, B]$. We now prove that m_B divides m_A. Let $b \in B$ and let $a \in A$ be such that $\pi(a) = b$. Denote by Z the centralizer of a and by Z_0 the centralizer of b. Note that $\mathcal{Z} := \pi^{-1}(Z_0)$ is a subgroup of A containing Z and that

$$|Z_0| = \frac{|\mathcal{Z}|}{|D|} = \frac{(\mathcal{Z} : Z)\,|Z|}{|D|}.$$

On the other hand, if we denote by C and C_0 the conjugacy classes of a and b in A and B respectively. We have

$$|C| = \frac{|A|}{|Z|} = \frac{|A|(\mathcal{Z} : Z)}{|D|\,|Z_0|} = (\mathcal{Z} : Z)\frac{|B|}{|Z_0|} = (\mathcal{Z} : Z)\,|C_0|.$$

Thus $|C_0|$ divides $|C|$ and therefore m_B divides m_A. ☐

Proof. Observe to start that, thanks to Theorem 3.1, the Integrality Conjecture for the extension K/k and the set of places S implies the Integrality Conjecture for the extension K'/k and the set of places S with $d_{G'}$ replaced by d_G. We first prove the result when $K = K'K^{\mathrm{ab}}$. In this situation, we shall actually prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ implies $\mathbf{BS}_{\mathrm{Gal}}(K'/k, S)$. Indeed, we have

[4] included in the statement of the Integrality Conjecture.

$d_G = d_{G'}$ by Lemma 4.4 since one can see, thanks to Lemma 4.5, that K is a strong central extension of K'/k. Let $\mathfrak{A}'$ be a fractional ideal of K'. We assume that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ holds, thus taking $\mathfrak{A} := \mathfrak{A}'\mathcal{O}_K$, we see that there exists an anti-unit α in K such that

$$\alpha\mathcal{O}_K = (\mathfrak{A}\mathcal{O}_K)^{d_G w_K \theta_{K/k,S}} = \mathfrak{A}'^{\,d_G w_K \theta_{K/k,S}}\mathcal{O}_K = \mathfrak{A}'^{\,d_G w_K \theta_{K'/k,S}}\mathcal{O}_K \tag{6.12}$$

where the last equality comes from Theorem 3.1, and such that $L := K(\gamma)$ is a strong central extension of K/k where $\gamma := \alpha^{1/w_K}$. Clearly, we have

$$\gamma\mathcal{O}_L = (\mathfrak{A}'\mathcal{O}_L)^{\,d_G\theta_{K'/k,S}}.$$

We now use Theorem 5.3(ii) with the extension L/K' and the element γ. The only assertion that needs to be checked is the fact that L is a strong central extension of K'/k. By Lemma 4.5, this is equivalent to the fact that $L = K'L^{\mathrm{ab}}$ where L^{ab} is the maximal sub-extension of L/k that is abelian over k. Clearly, $K^{\mathrm{ab}} \subset L^{\mathrm{ab}}$ thus $K'K^{\mathrm{ab}} = K \subset K'L^{\mathrm{ab}}$ and therefore $K'L^{\mathrm{ab}} = KL^{\mathrm{ab}} = L$, and L is a strong central extension of K'/k. Therefore $\mathbf{BS}_{\mathrm{Gal}}(K'/k,S;\mathfrak{A}')$ holds for all fractional ideals $\mathfrak{A}'$ of K' and thus $\mathbf{BS}_{\mathrm{Gal}}(K'/k,S)$ is satisfied.

We now prove the general case. By the first part, we can assume that K' contains K^{ab} and therefore, by hypothesis, w_K is relatively prime with the degree of K/K'. Let $\mathfrak{A}'$ be a fractional ideal of K'. Reasoning as above, we see that there exists $\alpha \in K^\circ$ such that

$$\alpha\mathcal{O}_K = \mathfrak{A}'^{\,d_G w_K \theta_{K'/k,S}}\mathcal{O}_K$$

and the extension L/K is a strong extension of K/k where $L := K(\gamma)$ and $\gamma := \alpha^{1/w_K}$. Denote by Γ the Galois group of L/k. For $\sigma \in \Gamma$, $L^\sigma = L$ is a Kummer extension of $K^\sigma = K$ generated by γ^σ. Thus there exist an integer n_σ relatively prime to w_K with $1 \leq n_\sigma \leq d := [L:K]$ and an element $\kappa_\sigma \in K^\times$ such that $\gamma^\sigma = \kappa_\sigma\gamma^{n_\sigma}$. Observe that, for δ an element of $\Delta := \mathrm{Gal}(L/K)$, we have $n_\delta = 1$ and κ_δ is a root of unity in K. Furthermore, using the fact that σ and δ commute, we get

$$\gamma^{\delta\sigma} = (\kappa_\sigma\gamma^{n_\sigma})^\delta = \kappa_\sigma\kappa_\delta^{n_\sigma}\gamma^{n_\sigma} = \gamma^{\sigma\delta} = (\kappa_\delta\gamma)^\sigma = \kappa_\delta^\sigma\kappa_\sigma\gamma^{n_\sigma}$$

and thus $\kappa_\delta^\sigma = \kappa_\delta^{n_\sigma}$. As δ runs through the elements of Δ, κ_δ runs through the roots of unity of order d, thus $\sigma - n_\sigma$ annihilates the group μ_d of d-th roots of unity. Assume now that σ lies in $A := \mathrm{Gal}(L/K')$. Therefore, σ fixes the group of roots of unity $\mu_{K'} = \mu_K$ and thus $n_\sigma = 1$. Using the fact that $\theta_{K'/k,S}$ is in the center of $\mathbb{C}[G]$, we get

$$\alpha^\sigma\mathcal{O}_K = (\mathfrak{A}'^\sigma)^{\,d_G w_K\theta_{K'/k,S}}\mathcal{O}_K = \mathfrak{A}'^{\,d_G w_K\theta_{K'/k,S}}\mathcal{O}_K = \alpha\mathcal{O}_K.$$

Since α is an anti-unit, there exists a root of unity ξ_σ in $K^\times$ such that $\alpha^\sigma = \xi_\sigma\alpha$. Combining with the above expression for γ^σ, we find that $\kappa_\sigma^{w_K} = \xi_\sigma$. Thus κ_σ is a root of unity in K and $\xi_\sigma = 1$. It follows that $\alpha \in K'$. Again we use Theorem 5.3(ii) to prove that $\tilde{\mathbf{BS}}_{\mathrm{Gal}}(K/k,S)$ holds for $\mathfrak{A}'$. It remains to prove that there is a strong central extension of K'/k containing γ. Let $L' := L^{\mathrm{ab}}K'$ where L^{ab} is the maximal sub-extension of L/k that is abelian over k. The Galois group of the extension L/L' is $[\Gamma,\Gamma]\cap A$. Hence, by Lemma 4.5, it is the maximal sub-extension of L/k that is strong central for K'/k. We now prove that $\gamma \in L'$. Denote by $\pi : \Gamma \to G$ the canonical surjection induced by the restriction to K. Its kernel is Δ, thus it restricts to an isomorphism between $[\Gamma,\Gamma]$ and $[G,G]$ (see also Lemma 4.3). Therefore $\gamma \in L'$ if and only if $\pi(\mathrm{Gal}(L/L')) \subset \pi(\mathrm{Gal}(L/K'(\gamma)))$, that is $\pi([\Gamma,\Gamma]\cap A) \subset \mathrm{Gal}(K/N)$ where $N = K \cap K'(\gamma)$. But N/K' is a sub-extension of K/K' of degree dividing w_K and therefore $N = K'$ and the above condition is always satisfied. Hence $\tilde{\mathbf{BS}}_{\mathrm{Gal}}(K'/k,S)$ holds and this concludes the proof. □

We conclude this section with a proof of when the validity of the conjecture is preserved when one enlarges the set S. Recall that, for $\chi \in \hat{G}$, we denote by ρ_χ an irreducible representation of G with character χ.

Lemma 6.7. *Let $\mathfrak{P}_1, \dots, \mathfrak{P}_t$ be prime ideals of K. We have*

$$\prod_{i=1}^{t} \sum_{\chi \in \hat{G}} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_i})) e_{\bar{\chi}} \in \frac{1}{|G|}\mathbb{Z}[G].$$

Proof. Let $\alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. It is easy to see that the above expression is invariant under the action of α using the fact that the map $\chi \mapsto \chi^\alpha$ is a bijection on $\hat{G}$. Therefore, it lies in $\mathbb{Q}[G]$. Now, by the orthogonality of characters, we have

$$\prod_{i=1}^{t} \sum_{\chi \in \hat{G}} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_i})) e_{\bar{\chi}} = \sum_{\chi \in \hat{G}} \prod_{i=1}^{t} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_i})) e_{\bar{\chi}}.$$

For all $\chi \in \hat{G}$, $|G|\, e_\chi$ and $\det(1 - \rho_\chi(\sigma_{\mathfrak{P}_i}))$, for $i = 1, \dots, t$, are algebraic integers and thus the result follows. $\square$

Proposition 6.8. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be distinct prime ideals of k not belonging to S. Define*

$$\omega := \prod_{i=1}^{t} \sum_{\chi \in \hat{G}} \det(1 - \rho_\chi(\sigma_{\mathfrak{P}_i})) e_{\bar{\chi}} \in \frac{1}{|G|}\mathbb{Z}[G]$$

where $\mathfrak{P}_i$ is a prime ideal of K above $\mathfrak{p}_i$, for $i = 1, \dots, t$. Let $d \geq 1$ be the smallest integer such that $d\omega \in \mathbb{Z}[G]$. Assume that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds and let $S' := S \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k, S'; \mathfrak{A})$ is satisfied for any fractional ideal $\mathfrak{A}$ of K whose class in Cl_K has order relatively prime to d.

Proof. Assume that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds. Let $\mathfrak{A}$ be an ideal of K whose class in Cl_K has order relatively prime to d. Thus there exists an ideal $\mathfrak{A}_0$ of K and $\eta \in K^\times$ such that $\mathfrak{A} = \eta \mathfrak{A}_0^d$. Let α_0 be an anti-unit of K such that

$$\alpha_0 \mathcal{O}_K = \mathfrak{A}_0^{d_G w_K \theta_{K/k,S}}$$

and the extension $K(\alpha_0^{1/w_K})$ is a strong central extension of K/k. Define

$$\alpha := \alpha_0^{d\omega} \eta^{d_G w_K \theta_{K/k,S'}}.$$

One checks directly that

$$\alpha \mathcal{O}_K = \mathfrak{A}^{d_G w_K \theta_{K/k,S'}}$$

From the proof of Propostion 6.1, we see that $\delta := \eta^{d_G w_K \theta_{K/k,S'}}$ is an anti-unit and that the extension $K(\delta^{1/w_K})$ is a strong central extension of K/k. Therefore, α is an anti-unit and the extension $K(\alpha^{1/w_K}) \subset K(\alpha_0^{1/w_K}, \delta^{1/w_K})$ is a strong central extension of K/k by Lemma 4.6. Thus $\mathbf{BS}_{\mathrm{Gal}}(K/k, S')$ holds. $\square$

7. Groups with a normal abelian subgroup of prime index

In this final section, we study the conjecture in the case where the Galois group G contains an abelian normal subgroup H such that the index $(G : H)$ is equal to a prime number ℓ. We assume furthermore that G is not abelian. We prove in this setting that the Galois Brumer-Stark conjecture is satisfied provided the abelian Brumer-Stark conjecture holds for the abelian extensions K^{ab}/k and K/K^H.

Let m denote the order of H, thus $|G| = m\ell$. We have $[G,G] \subset H$ since G/H is cyclic of order ℓ and therefore K^H is a subfield of K^{ab}. Let S_H denote the set of places in K^H that are above the places in S. The set S_H contains the infinite places of K^H and the finite places that ramify in K/K^H. The first result of this section gives a decomposition of the Brumer-Stickelberger element in this situation.

Theorem 7.1. *Let* $N_{[G,G]} := \sum_{c\in[G,G]} c \in \mathbb{Z}[G]$. *We have*

$$\theta_{K/k,S} = \nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S}) + \Big(1 - \frac{1}{s_G} N_{[G,G]}\Big)\theta_{K/K^H,S_H}.$$

Proof. By (3.10), it remains to prove that $\theta^{(>1)}_{K/k,s}$ is equal to $\big(1-\frac{1}{s_G}N_{[G,G]}\big)\theta_{K/K^H,S_H}$. The group G contains an abelian normal subgroup of index ℓ, thus the degrees of the irreducible characters of G divide ℓ. Hence any character in $\hat{G}$ such that $\chi(1) > 1$ is of degree ℓ. Denote by $\hat{G}_\ell$ the set of irreducible characters of G of degree ℓ.

Lemma 7.2. *Let $\hat{H}_\ell$ be the set of irreducible characters of H whose kernel does not contain $[G,G]$. For $\chi \in \hat{G}_\ell$, define $\hat{H}_\ell(\chi)$ to be the subset of those characters in $\hat{H}_\ell$ whose induction to G is χ. Then, for all $\chi \in \hat{G}_\ell$ and $g \in G$, we have*

$$\chi(g) = \begin{cases} 0 & \text{if } g \notin H, \\ \sum\limits_{\varphi\in\hat{H}_\ell(\chi)} \varphi(g) & \text{if } g \in H, \end{cases}$$

and

$$\hat{H}_\ell = \bigcup_{\chi\in\hat{G}_\ell} \hat{H}_\ell(\chi) \quad (\textit{disjoint union}).$$

Furthermore, each $\hat{H}_\ell(\chi)$ has ℓ elements.

Proof of the lemma. Let φ be a character in $\hat{H}_\ell$ and let $\chi := \mathrm{Ind}_H^G(\varphi)$. Then χ is of degree ℓ. Assume χ is not irreducible. Then it is a sum of ℓ degree-1 characters and all these characters are trivial on $[G,G]$. By Frobenius reciprocity law, the restriction of any of these characters to H is equal to φ. Thus φ is trivial on $[G,G]$, a contradiction. Therefore χ is irreducible and lies in $\hat{G}_\ell$. The restriction of χ to H is the sum of ℓ characters of H, and using, once again, Frobenius reciprocity law, we see that these characters are exactly the characters of H whose induction to G is χ and that there are all distinct. Therefore, we have proved that, if $\chi \in \hat{G}_\ell$ is the induction of some character in $\hat{H}_\ell$, then the set $\hat{H}_\ell(\chi)$ contains ℓ distinct characters, say $\varphi_1, \dots, \varphi_\ell$, such that $\chi_{|H} = \varphi_1 + \cdots + \varphi_\ell$. Furthermore, if χ' is another character of $\hat{G}_\ell$ induced from a character in $\hat{H}_\ell$, the sets $\hat{H}_\ell(\chi)$ and $\hat{H}_\ell(\chi')$ are clearly disjoint. This implies that $\hat{H}_\ell$ is the disjoint union of the $\hat{H}_\ell(\chi)$'s for $\chi \in \hat{G}_\ell$. We now prove that $\hat{H}_\ell(\chi)$ is non-empty for all $\chi \in \hat{G}_\ell$. This amounts to prove that any character in $\hat{G}_\ell$ is the induction of some character in $\hat{H}_\ell$. Characters of H whose kernel contains $[G,G]$ are in bijection with characters of $H/[G,G]$. Denote by t the index of $[G,G]$ in H. The number of characters in $\hat{H}_\ell$ is therefore $m-t$ and therefore, by the above discussion, the inductions of characters in $\hat{H}_\ell$ yield $(m-t)/\ell$ characters in $\hat{G}_\ell$. On the other hand, we have the formula $m\ell = t\ell + a\ell^2$, where a is the number of characters in $\hat{G}_\ell$, which is obtained by looking at the degree of the irreducible characters of G and using the fact that $(G : [G,G]) = t\ell$. Therefore, we have $a = (m-t)/\ell$ and all the characters of $\hat{G}_\ell$ are inductions of characters in $\hat{H}_\ell$. To conclude, it remains to prove the expression for $\chi \in \hat{G}_\ell$. Let $\varphi \in \hat{H}_\ell(\chi)$. Recall the expression of χ in terms of φ; for all $g \in G$, we have

$$\chi(g) = \frac{1}{m} \sum_{\substack{r\in G \\ rgr^{-1}\in H}} \varphi(rgr^{-1}).$$

Since the group H is normal in G, $rgr^{-1} \in H$ if and only $g \in H$. Thus $\chi(g) = 0$ if $g \notin H$. If $g \in H$, the expression follows from the fact that $\chi_{|H} = \sum_{\varphi\in\hat{H}_\ell(\chi)} \varphi$. ☐

As a consequence of the above lemma, we have, for $\chi \in \hat{G}_\ell$,

$$e_\chi = \sum_{\varphi \in \hat{H}_\ell(\chi)} e_\varphi$$

where e_φ is the idempotent of $\mathbb{C}[H]$ associated to the character φ. We now compute

$$\begin{aligned}
\theta^{(>1)}_{K/k,s} &= \sum_{\chi \in \hat{G}_\ell} L_{K/k,S}(0,\chi)\, e_{\bar\chi} = \sum_{\chi \in \hat{G}_\ell} L_{K/k,S}(0,\chi) \sum_{\varphi \in \hat{H}_\ell(\chi)} e_{\bar\varphi} \\
&= \sum_{\chi \in \hat{G}_\ell} \sum_{\varphi \in \hat{H}_\ell(\chi)} L_{K/k,S}(0, \operatorname{Ind}_H^G \varphi)\, e_{\bar\varphi} = \sum_{\chi \in \hat{G}_\ell} \sum_{\varphi \in \hat{H}_\ell(\chi)} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} \\
&= \sum_{\varphi \in \hat{H}_\ell} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} = \sum_{\varphi \in \hat{H}} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} - \sum_{\varphi \in \hat{H} \setminus \hat{H}_\ell} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} \\
&= \theta_{K/K^H,S_H} - \sum_{\substack{\varphi \in \hat{H} \\ [G,G] \subset \operatorname{Ker} \varphi}} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi}.
\end{aligned}$$

Let φ be a character of H whose kernel contains $[G,G]$. Let $\tilde\varphi$ be the only character of $J := H/[G,G]$ such that the inflation of $\tilde\varphi$ to H is equal to φ. From the properties of Artin L-function, we have $L_{K/K^H,S_H}(0,\varphi) = L_{K^{\mathrm{ab}}/K^H,S_H}(0,\tilde\varphi)$ and a direct calculation shows that $e_\varphi = \nu_H^{\mathrm{ab}}(e_{\tilde\varphi})$ where $e_{\tilde\varphi}$ is the idempotent of $\mathbb{C}[G^{\mathrm{ab}}]$ associated to $\tilde\varphi$, $\nu_H^{\mathrm{ab}} : \mathbb{C}[J] \to \mathbb{C}[H]$ is the map defined for $\tilde g \in J$ by

$$\nu_H^{\mathrm{ab}}(\tilde g) := \frac{1}{s_G} \sum_{\pi_H^{\mathrm{ab}}(g) = \tilde g} g,$$

and extended linearly to $\mathbb{C}[J]$, and $\pi_H^{\mathrm{ab}} : H \to J$ is the canonical surjection. Therefore,

$$\begin{aligned}
\sum_{\substack{\varphi \in \hat{H} \\ [G,G] \subset \operatorname{Ker} \varphi}} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} &= \sum_{\tilde\varphi \in \hat{J}} L_{K^{\mathrm{ab}}/K^H,S_H}(0,\tilde\varphi) \nu_H^{\mathrm{ab}}(e_{\bar{\tilde\varphi}}) \\
&= \nu_H^{\mathrm{ab}}\Big(\sum_{\tilde\varphi \in \hat{J}} L_{K^{\mathrm{ab}}/K^H,S_H}(0,\tilde\varphi)\, e_{\bar{\tilde\varphi}} \Big) \\
&= \nu_H^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/K^H,S_H}).
\end{aligned}$$

Now, for $\alpha \in \mathbb{C}[H]$ and $\beta \in \mathbb{C}[J]$, one checks readily that $\alpha \nu_H^{\mathrm{ab}}(\beta) = \nu_H^{\mathrm{ab}}(\tilde\alpha\beta)$ where $\tilde\alpha := \pi_H^{\mathrm{ab}}(\alpha)$. Therefore, we have

$$\nu_H^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/K^H,S_H}) = \theta_{K/K^H,S_H}\, \nu_H^{\mathrm{ab}}(1) = \frac{1}{s_G} N_{[G,G]}\, \theta_{K/K^H,S_H}.$$

The result then follows by substituting in the above expression. □

The main advantage of the decomposition given by the theorem is the fact that the extensions involved are abelian. Therefore, in our study of $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ in that setting, we can reduce to the abelian case. As a first consequence, we prove that the Integrality Conjecture is satisfied in this situation.

Proposition 7.3. *We have*

$$(s_G - N_{[G,G]})\theta_{K/K^H,S_H} \in \mathbb{Z}[G].$$

In particular, for almost all prime ideals $\mathfrak{P}$ of K, $d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\,\theta_{K/k,S} \in \mathbb{Z}[G]$ where $\mathfrak{p}$ is the prime ideal of k below $\mathfrak{P}$.

Proof. First note that, by Theorem 7.1 and the discussion after (3.10), the first assertion implies the second assertion. Now, we have

$$(s_G - N_{[G,G]})\theta_{K/K^H,S_H} = \sum_{c\in[G,G]} (1-c)\theta_{K/K^H,S_H}.$$

But $1-c \in \mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$ for all $c \in [G,G]$ and thus, by the properties of the abelian Brumer-Stickelberger element, all the terms in that last sum are in $\mathbb{Z}[H]$. The first assertion and the proof of the proposition follow. $\square$

We now prove that the Galois Brumer-Stark conjecture in that setting is a consequence of the abelian Brumer-Stark conjecture.

Theorem 7.4. *Assume that* $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ *and* $\mathbf{BS}(K/K^H,S_H)$ *hold. Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ *is satisfied.*

Proof. We will prove the theorem using Proposition 6.3. First note that, by Theorem 7.1, we have

$$\theta^{(>1)}_{K/k,S} = \frac{1}{s_G}(s_G - N_{[G,G]})\,\theta_{K/K^H,S_H}.$$

Let $\mathfrak{A}$ be a fractional ideal of K. By our hypothesis, there exist $\alpha_0 \in K^\circ$ such that

$$\mathfrak{A}^{w_K\theta_{K/K^H,S_H}} = \alpha_0\mathcal{O}_K$$

and the extension $K(\gamma_0)/K^H$ is abelian where $\gamma_0 := \alpha_0^{1/w_K}$. Define

$$\beta := \alpha_0^{(d_G/s_G)(s_G-N_{[G,G]})} = \Big(\prod_{c\in[G,G]} \alpha_0^{1-c}\Big)^{d_G/s_G}.$$

By construction, β is an anti-unit of K and satisfies

$$\beta\mathcal{O}_K = \mathfrak{A}^{d_G w_K \theta^{(>1)}_{K/k,S}}.$$

It remains to prove that $K(\beta^{1/w_K})$ is a strong central extension of K/k. We will actually prove that $K(\beta^{1/w_K}) = K$. Let L_0 be the Galois closure of $K(\gamma_0)/k$. Denote by Γ_0 the Galois group $\mathrm{Gal}(L_0/k)$. Let $c_0 \in [\Gamma_0,\Gamma_0]$. Note that $c_0 \in \mathrm{Gal}(L_0/K^H)$ since K^H/k is abelian. Thus, by Theorem 2.2, there exists a prime ideal $\mathfrak{P}_0$ of L_0, relatively prime to the order of μ_{L_0}, whose Frobenius automorphism in Γ_0 is equal to c_0, and an anti-unit $\alpha_{0,\mathfrak{p}_H} \in K^\circ$ such that $\alpha_{0,\mathfrak{p}_H} \equiv 1 \pmod{^*\ \mathfrak{p}_H\mathcal{O}_K}$ and

$$\alpha_{0,\mathfrak{p}_H}\mathcal{O}_K = \mathfrak{A}^{(\sigma_{\mathfrak{p}_H} - \mathcal{N}(\mathfrak{p}_H))\theta_{K/K^H,S_H}}$$

where $\mathfrak{p}_H$ is the prime ideal of K^H below $\mathfrak{P}_0$ and $\sigma_{\mathfrak{p}_H}$ is the Frobenius automorphism of $\mathfrak{p}_H$ in H. We have

$$\begin{aligned}\gamma_0^{c_0-1}\mathcal{O}_{L_0} &= \mathfrak{A}^{(c_0-1)\theta_{K/K^H,S_H}}\mathcal{O}_{L_0}\\ &= \mathfrak{A}^{(\sigma_{\mathfrak{p}_H}-\mathcal{N}(\mathfrak{p}_H))\theta_{K/K^H,S_H}}\mathfrak{A}^{(\mathcal{N}(\mathfrak{p}_H)-1)\theta_{K/K^H,S_H}}\mathcal{O}_{L_0}\\ &= \alpha_{0,\mathfrak{p}_H}\alpha_0^{(\mathcal{N}(\mathfrak{p}_H)-1)/w_K}\mathcal{O}_{L_0}.\end{aligned}$$

Observe that γ_0, $\alpha_{0,\mathfrak{p}_H}$ and α_0 are anti-units, thus there exists a root of unity $\xi \in \mu_{L_0}$ such that $\xi\gamma_0^{c_0-1} = \alpha_{0,\mathfrak{p}_H}\alpha_0^{(\mathcal{N}(\mathfrak{p}_H)-1)/w_K}$ and the latter belongs to K° since w_K divides $\mathcal{N}(\mathfrak{p}_H)-1$. Raising to the power w_K, we get

$$\xi^{w_K}\alpha_0^{\sigma_{\mathfrak{p}_H}-1} = \alpha_{0,\mathfrak{p}_H}^{w_K}\alpha_0^{\mathcal{N}(\mathfrak{p}_H)-1}$$

and therefore

$$\xi^{w_K} \equiv \alpha_0^{\mathcal{N}(\mathfrak{p}_H)-\sigma_{\mathfrak{p}_H}} \equiv 1 \pmod{^*\ \mathfrak{p}_H\mathcal{O}_K}.$$

Therefore we find that $\xi^{w_K} = 1$, hence $\xi \in \mu_K$ and $\gamma_0^{c_0-1} \in K$.

Now, for all $c \in [G, G]$, fix an element c_0 in $[\Gamma_0, \Gamma_0]$ whose restriction to K is equal to c, and define

$$\delta := \Big(\prod_{c \in [G,G]} \gamma_0^{1-c_0} \Big)^{d_G/s_G}.$$

By the above computation, we see that $\delta \in K$ and, by construction, we get that $\delta^{w_K} = \beta$. Therefore $K(\beta^{1/w_K}) = K$ and the result follows. $\square$

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